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en mathématique et physique

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Odd Khovanov Homology & Higher Representation Theory

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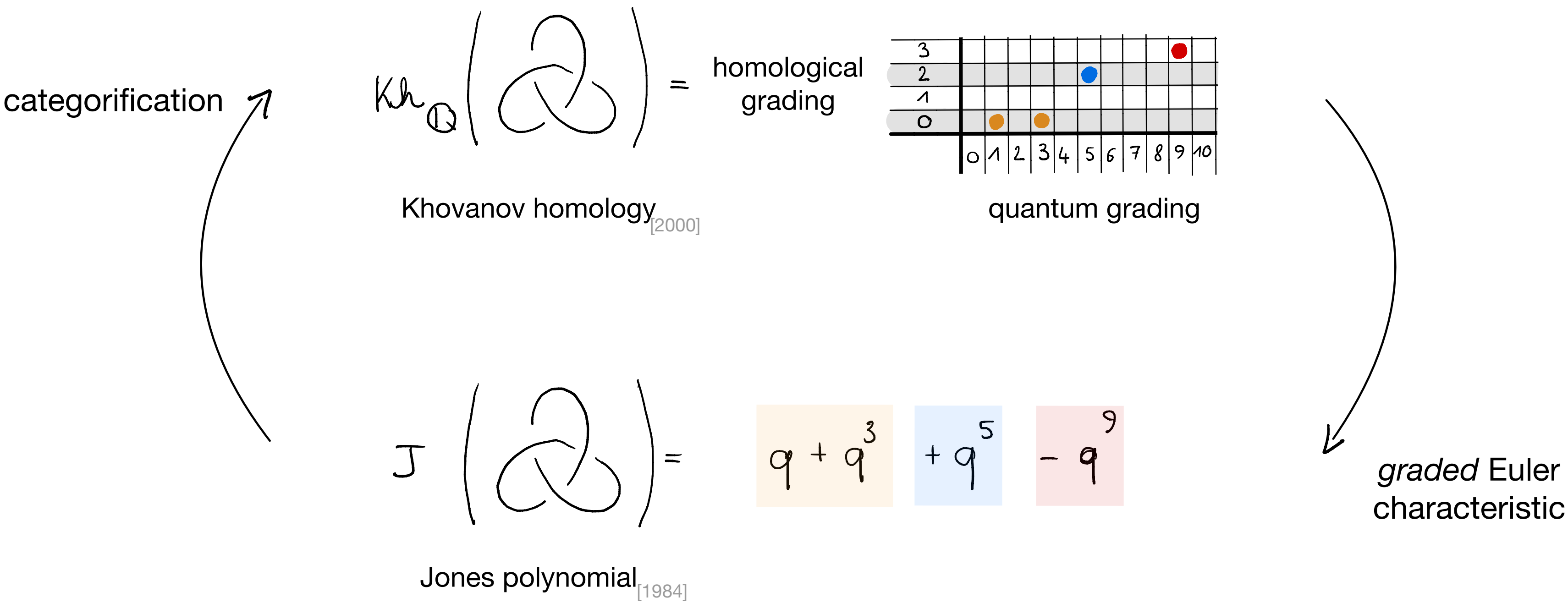
(joint work with Pedro Vaz)

EQuAL seminar — May 23, 2024

Overview



A **stronger** invariant, but above all, a **functorial** one! ⇛ **can see the fourth dimension!**

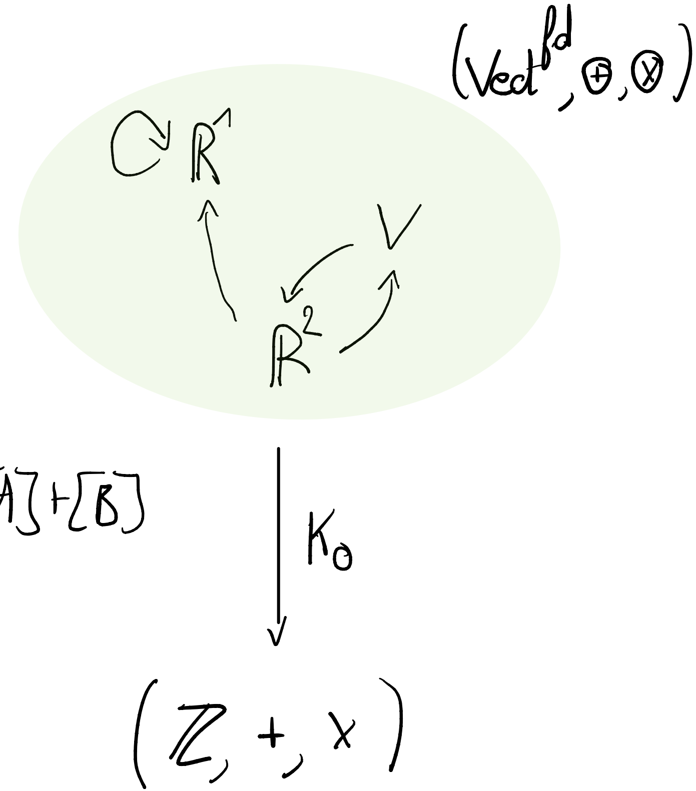
Categorification

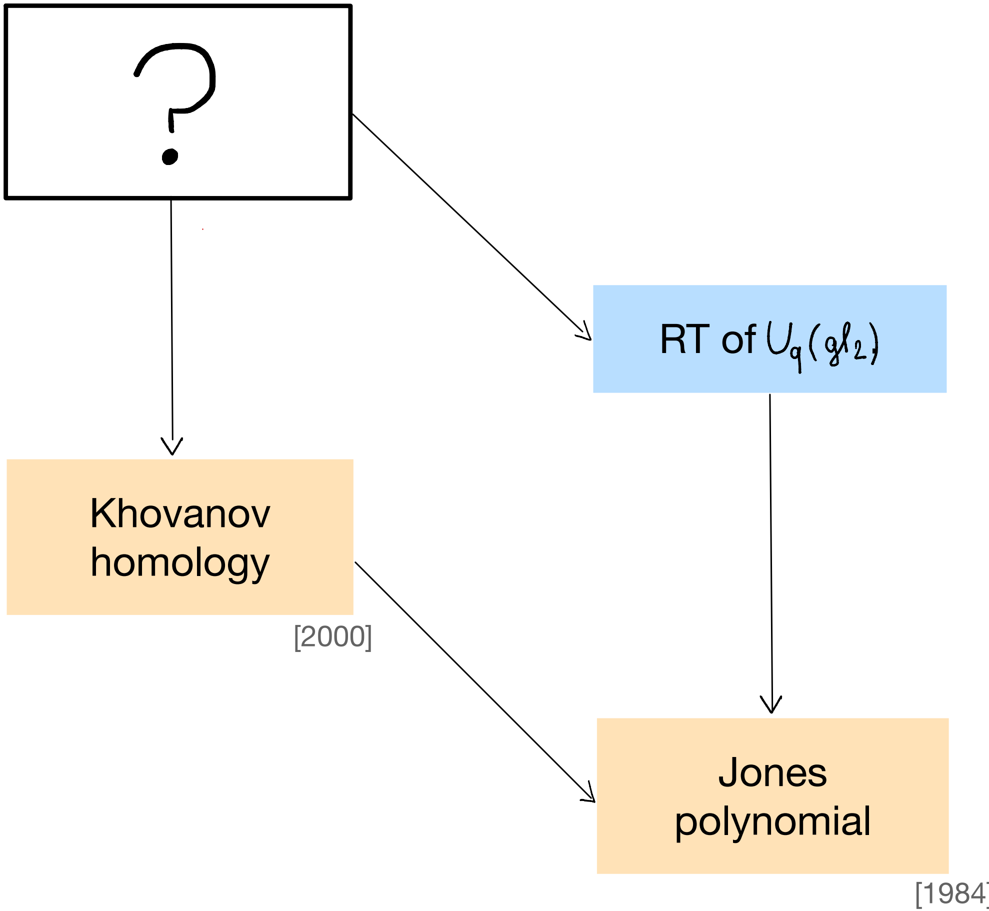
"add an extra layer"

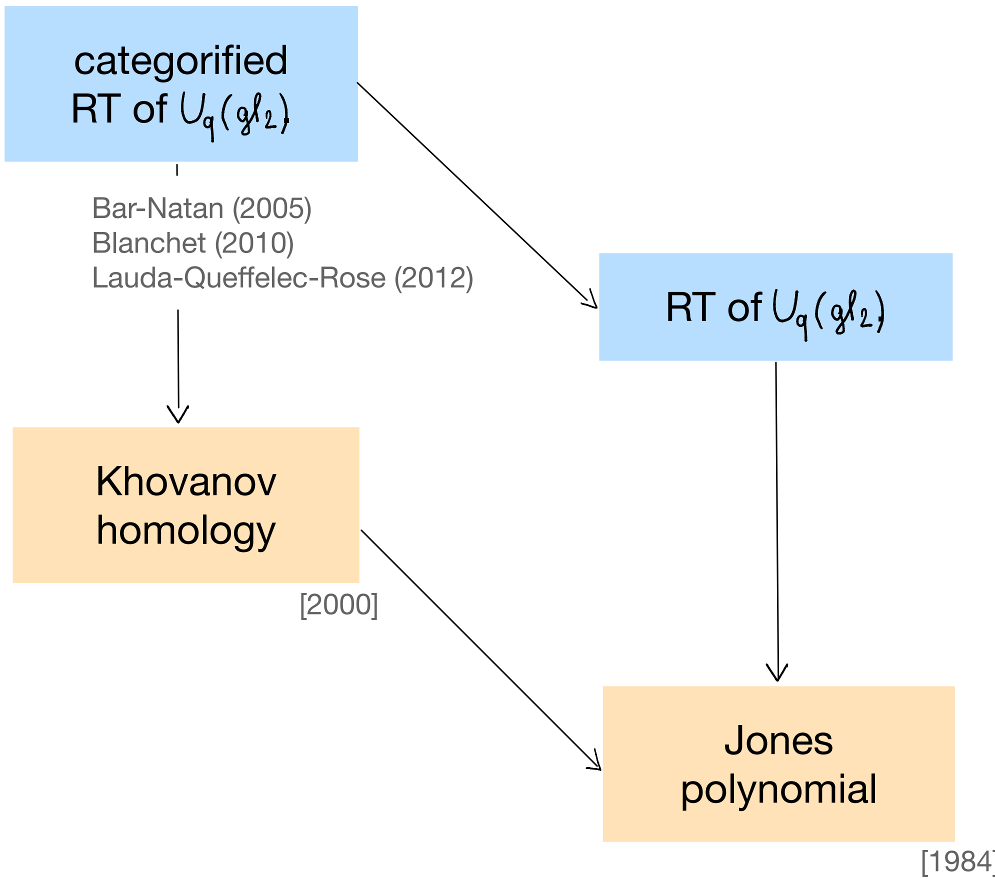
$\mathcal{C}$ additive monoidal:

$K_0(\mathcal{C}) = \mathbb{Z}[[A] \text{ iso classes of obj.}]$ / $[A \oplus B] = [A] + [B]$

$[A] \cdot [B] := [A \otimes B]$

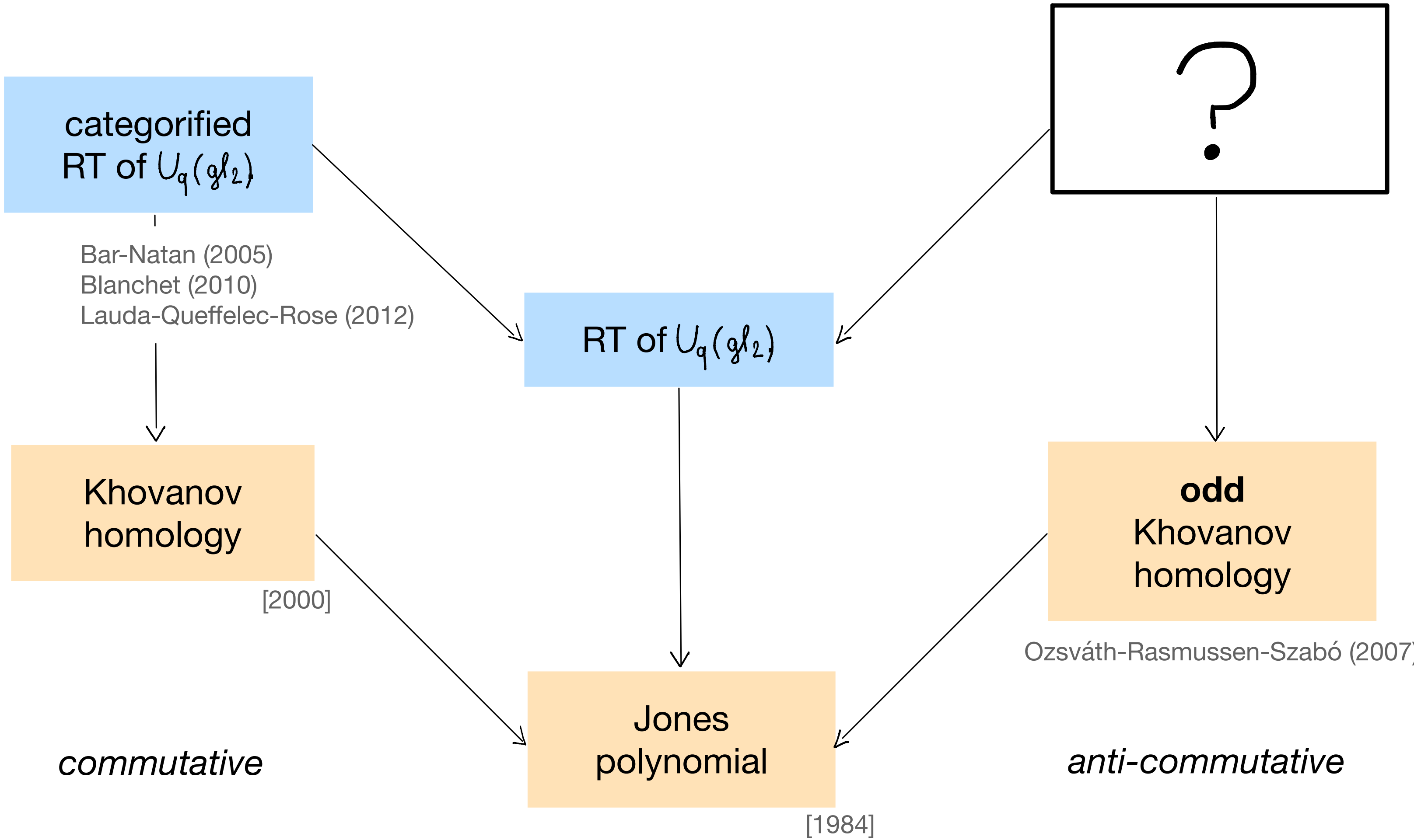



?
RT of $U_q(\mathfrak{gl}_2)$
Khovanov homology
[2000]
Jones polynomial
[1984]

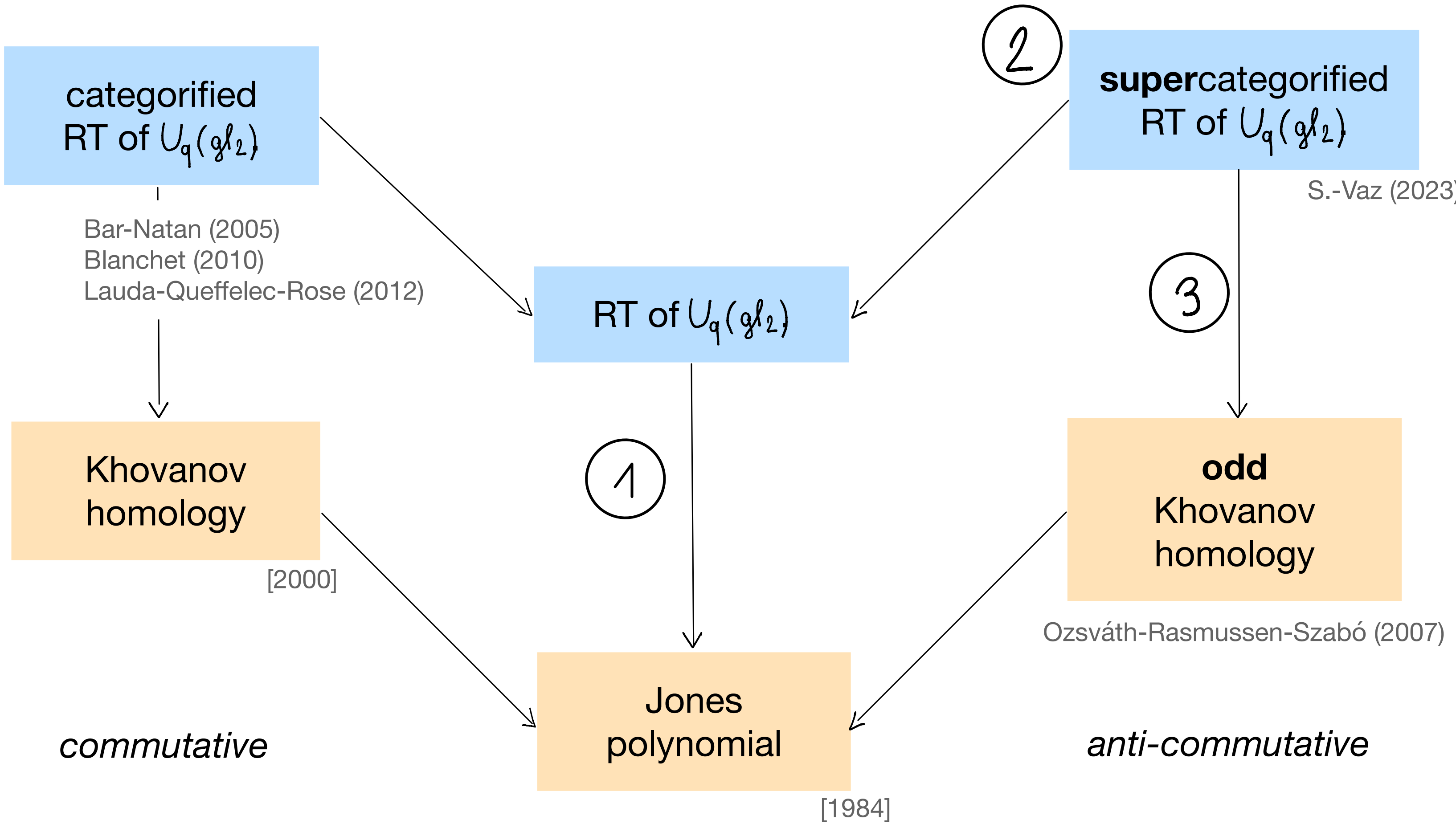


=> gives a conceptual framework

- locality (extension to tangles)
- functoriality



categorified RT of $U_q(\mathfrak{gl}_2)$
?
Bar-Natan (2005)
Blanchet (2010)
Lauda-Queffelec-Rose (2012)
RT of $U_q(\mathfrak{gl}_2)$
Khovanov homology
[2000]
odd Khovanov homology
Ozsváth-Rasmussen-Szabó (2007)
Jones polynomial
[1984]
commutative
anti-commutative



categorified RT of $U_q(\mathfrak{gl}_2)$
Bar-Natan (2005)
Blanchet (2010)
Lauda-Queffelec-Rose (2012)
RT of $U_q(\mathfrak{gl}_2)$
supercategorified RT of $U_q(\mathfrak{gl}_2)$
S.-Vaz (2023)
1
2
3
Khovanov homology
[2000]
Jones polynomial
[1984]
odd Khovanov homology
Ozsváth-Rasmussen-Szabó (2007)
commutative
anti-commutative

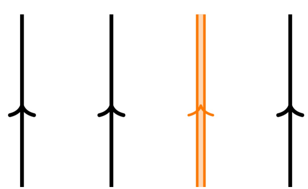
① RT of the Jones polynomial

Fact:

There exists a $\mathbb{Z}[q,q^{-1}]$-linear monoidal category $\mathcal{W}eb$ such that:

$$\mathcal{W}eb \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong \mathrm{Fund}(U_q(\mathfrak{gl}_2))$$

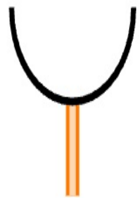
- objects are sequences of  and  :

 $\longleftrightarrow$ $V \otimes V \otimes \Lambda^2(\mathbb{C}) \otimes V$

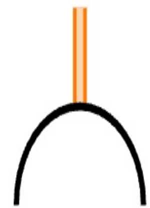
$V = \mathbb{C}(q)^2$ standard rep

determinant rep

- morphisms are generated by:

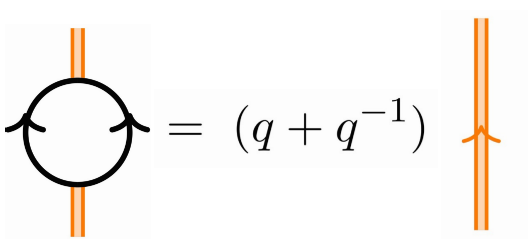


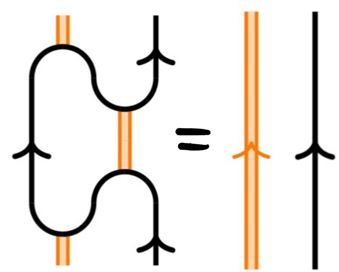
and

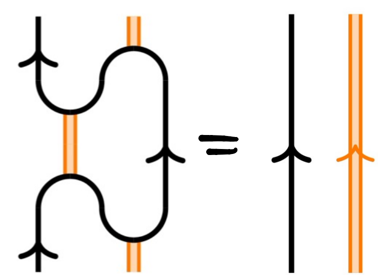


$\longleftrightarrow$ $V\otimes V \to \Lambda^2(V)$, $\quad v_1\otimes v_2 \mapsto v_1\wedge v_2$

modulo relations:



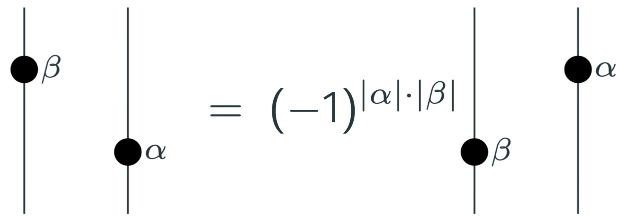




② Categorified RT of $U_q(\mathfrak{gl}_2)$

Definition: *2-supercategory* = ~ 2-category, but…

2-morphisms have parities and **interchange law is twisted:**

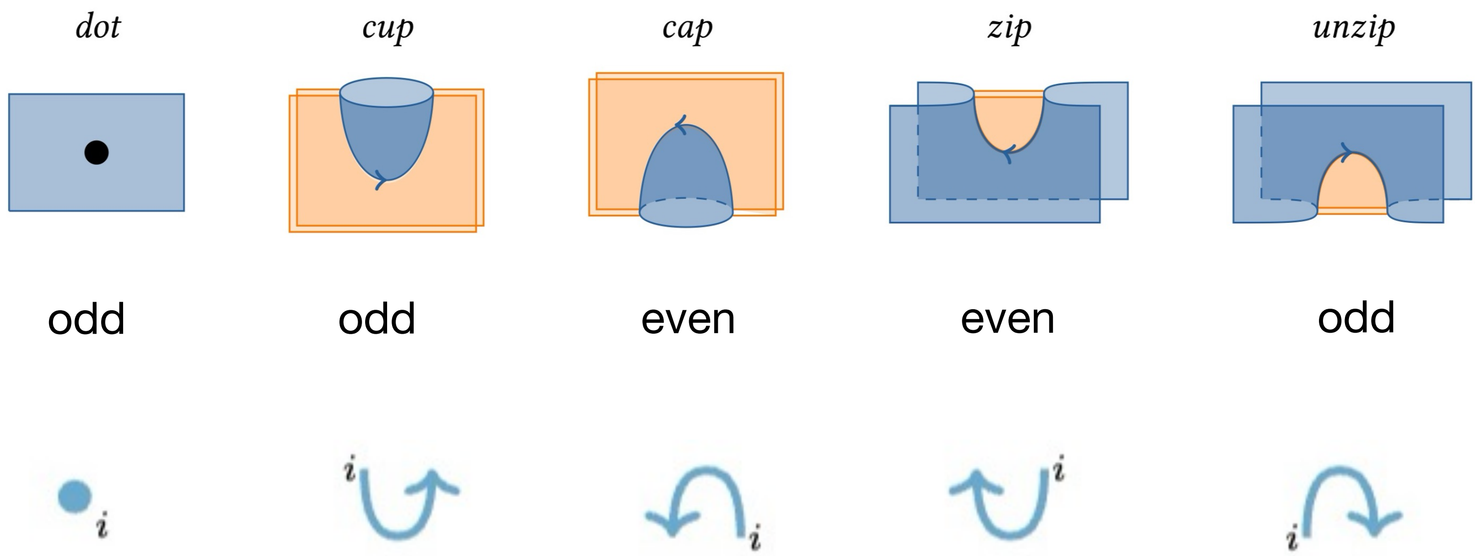


Theorem (S.-Vaz 2023, S. to appear)

∃ a 2-*super*category **SFOAM** s.t.:

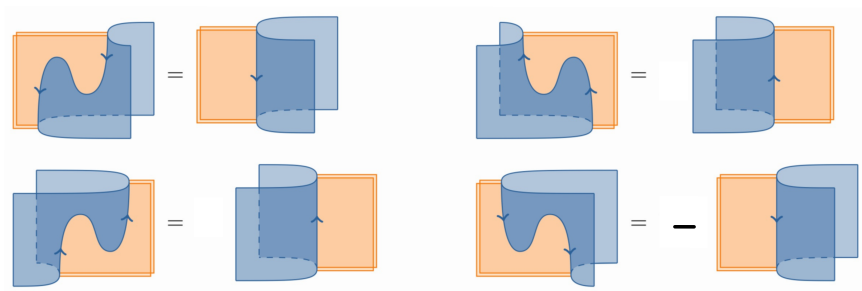
$$K_0(\mathrm{SFOAM}) \cong \mathrm{Web}$$

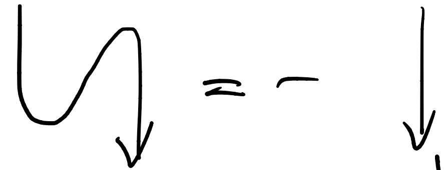
- generators:



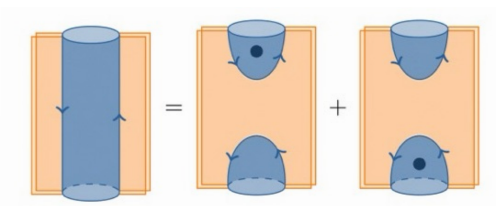
⇝ diagrammatics (akin to singular Soergel bimodules)

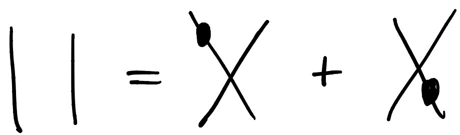
- Relations:





super adjunction!



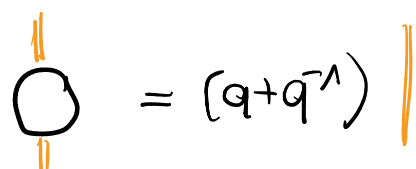


odd nil Hecke algebra

$\longleftrightarrow$ $1 = x_i \partial_i + \partial_i x_{i+1}$

odd symmetric functions

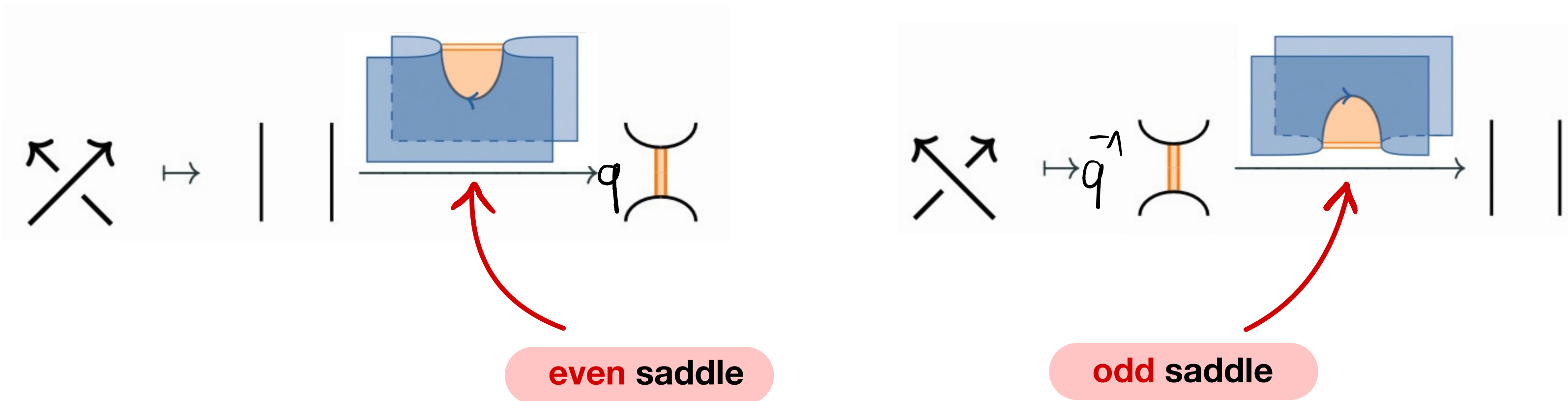
AND categorifies

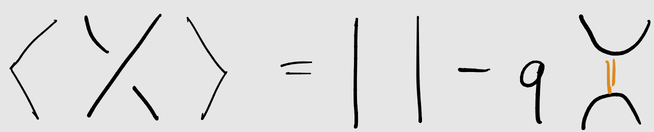


and more...

③ The RT of odd Khovanov homology

invariant of tangles defines **as a tensor product** of complexes in **SFOAM**:





categorifies the Kauffman bracket

Theorem (S. 2020)

In any 2-supercategory, $\exists$ a tensor product of complexes s.t.:

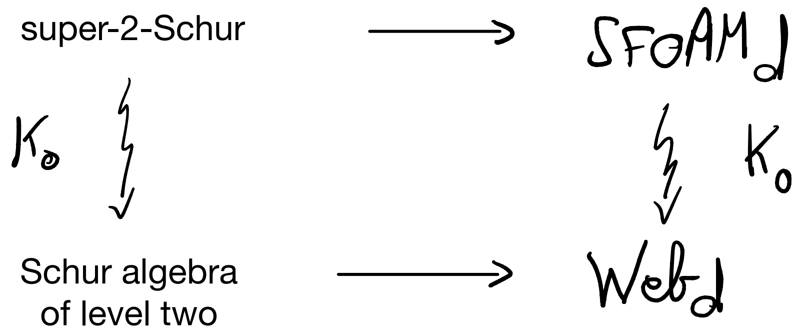
$$A^\bullet \simeq B^\bullet \quad \text{and} \quad C^\bullet \simeq D^\bullet \quad \Rightarrow \quad A^\bullet \otimes C^\bullet \simeq B^\bullet \otimes D^\bullet$$

Theorem (S.-Vaz, 2023)

- Its homotopy type is an invariant of tangles
- It **coincides with odd Khovanov homology** when restricted to links

original construction	new construction
ad-hoc sign fixes	*signs controlled by* ***parities***

Experts' corner

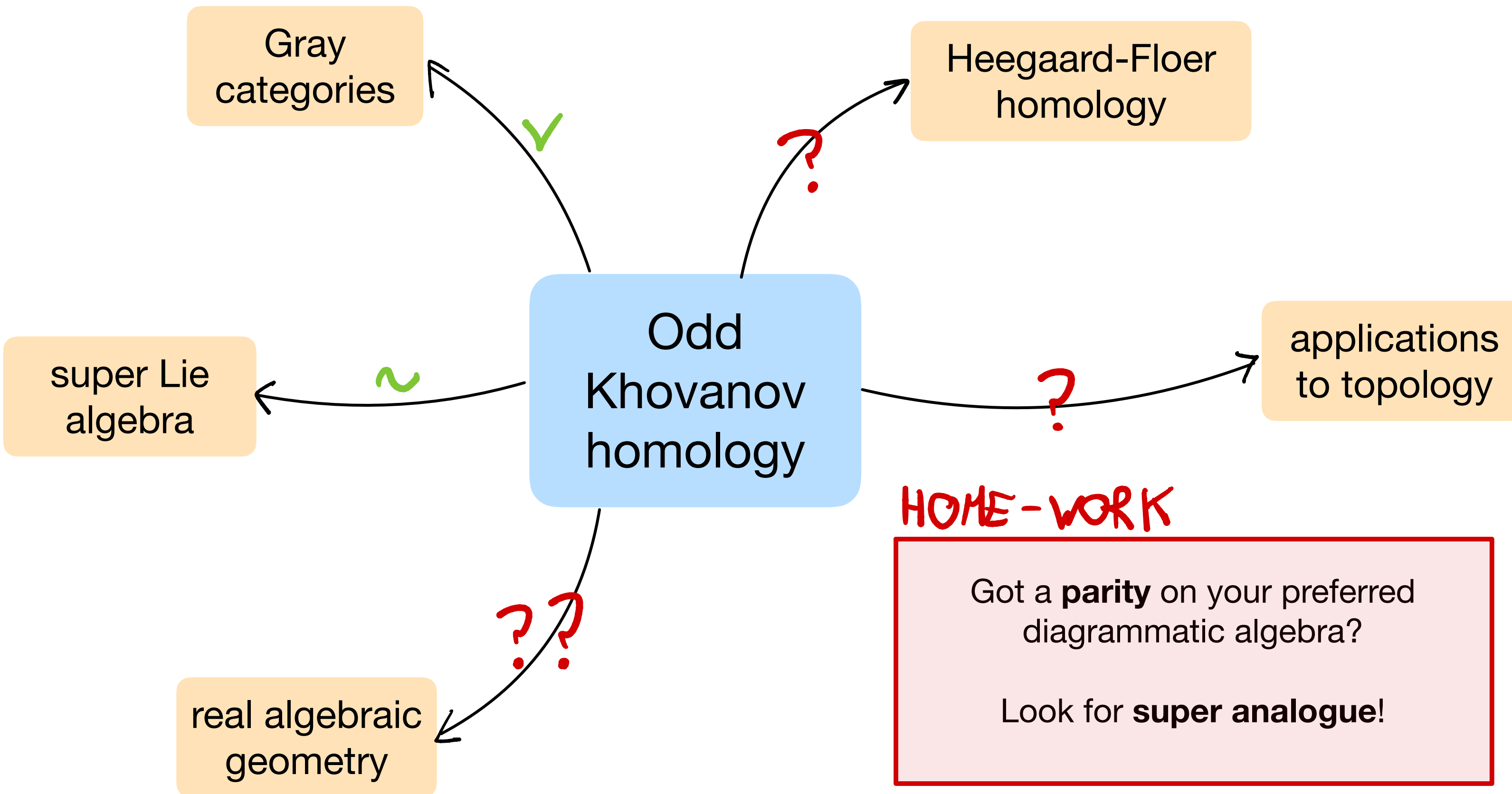


super-2-categories $\subset$ **graded**-2-categories $\subset$ linear **Gray** categories

universal tricategorical model

How to show that it doesn't degenerate? **linear Gray rewriting theory**

Perspective



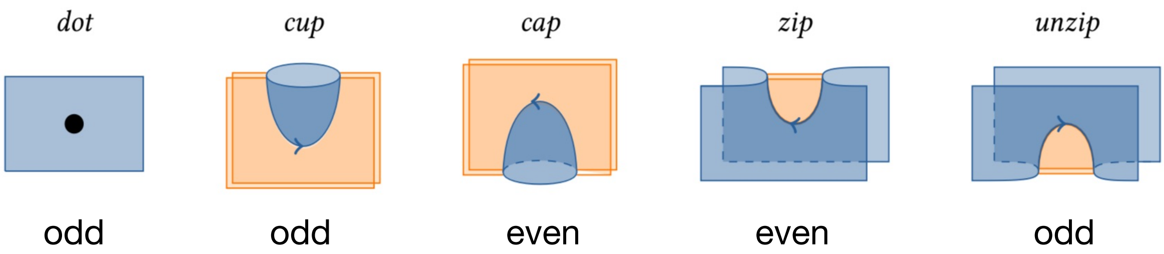
HOME-WORK

Got a **parity** on your preferred diagrammatic algebra?

Look for **super analogue**!

Presentation of the 2-supercategory **SFOAM**

G
E
N
E
R
A
T
O
R
S



R
E
L
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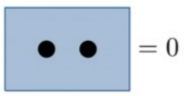
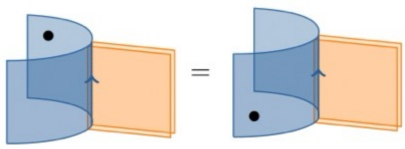
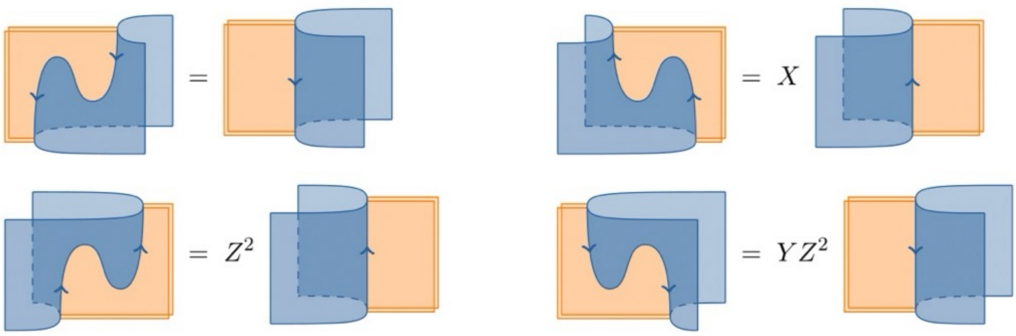


dot annihilation

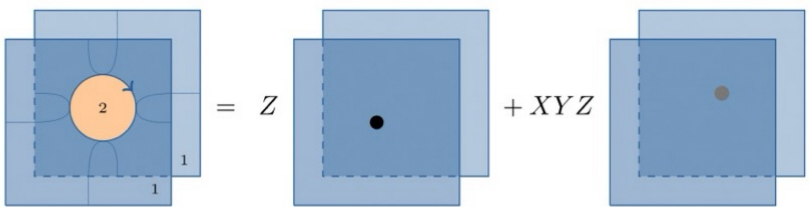


dot migration

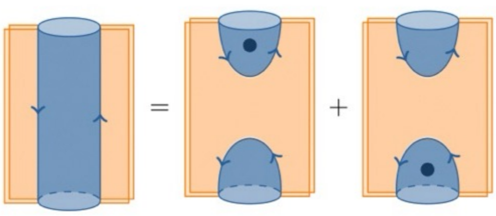


zigzag relations

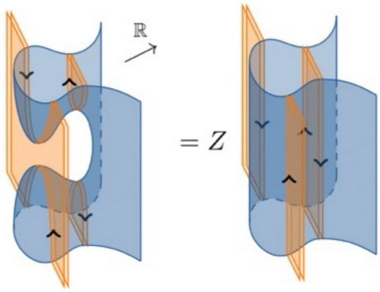
bubble evaluations



horizontal neck-cutting relation



vertical neck-cutting relation



squeezing relation